Transition from two-component 332 Halperin state to one-component Jain state at filling factor $\frac{2}{5}$

M. V. Milovanović[1] and Z. Papić[1,2]
[1]*Scientific Computing Laboratory, Institute of Physics, University of Belgrade, P.O. Box 68, 11 000 Belgrade, Serbia*
[2]*Laboratoire de Physique des Solides, Université Paris-Sud–CNRS, UMR 8502, F-91405 Orsay Cedex, France*
(Received 18 March 2010; revised manuscript received 27 June 2010; published 27 July 2010)

We study the transition induced by tunneling from the two-component 332 Halperin's state to the one-component Jain's state at the filling factor $\nu=2/5$. In exact diagonalizations of small systems two possibilities for the transition are found: (a) avoided level crossing, and (b) level crossing, i.e., first-order transition in the case of Coulomb interaction and short range interaction, respectively. An effective bosonic model with p-wave pairing for the transition is proposed. The relevance of the Gaffnian state for the transition is discussed as well as possible consequences of our model on the effective description of the Jain's state.

DOI: 10.1103/PhysRevB.82.035316 PACS number(s): 73.43.Cd, 73.21.Fg, 71.10.Pm

I. INTRODUCTION

Topological phases of matter[1] find their concrete realizations in quantum Hall physics within the systems of two-dimensional (2D) electrons in high-magnetic fields. They are characterized by a gap to all excitations and by degeneracy of the ground state on higher genus surfaces. By changing a parameter of the electronic system, we may induce a quantum phase transition from one topological phase to the other. Due to their nature and discrete characterization, we expect that the system gap closes at the transition between topological phases that differ in the topological invariants, i.e., the numbers that characterize them.

The fractional quantum Hall (FQH) states can be characterized by the filling factor ν i.e. particular ratio between the density of electrons and the strength of the magnetic field at which they appear. In spin-polarized systems, a successful explanation of various FQH states at different filling factors is given by Jain's states.[2] On the other hand, for the same filling factors we may have states with two or more different species i.e. the Halperin states.[3] Usually for a given filling factor described by the Jain's state, the corresponding Halperin state has the same vacuum degeneracy but some other characteristic numbers may differ. By applying tunneling to a two-component Halperin state we may transform this state into the one-component (spin-polarized) Jain's state. Tunneling as a perturbation that drives the transition from the two-component to a one-component FQH system was studied previously by analytical[4,5] and numerical[6] means.

In this paper we study the transition from Halperin's two-component 332 state to the one-component Jain state at the filling factor $\nu=2/5$ via tunneling. The interest is threefold: we would like to find out (a) about the nature of quantum phase transitions between topological phases which are similar (332 and Jain's state have the same ground state degeneracy[7–9] but different shift[10,11]), (b) we would like to find if Gaffnian[12] can be characterized as a critical state in these circumstances when the gap closes, and (c) we explore possible consequences for the effective description of the Jain state due to a better understanding of the transition. In Sec. II, we define the electronic system that we consider. Section III contains the results of the exact diagonalization studies of the transition. In Sec. IV, we introduce an effective bosonic model of the system and the transition induced by tunneling. Section V is devoted to conclusions.

II. SYSTEM UNDER CONSIDERATION

We consider the quantum Hall bilayer in the presence of the vector potential $\mathbf{A}$ that describes a strong magnetic field, $B\hat{z}=\nabla\times\mathbf{A}$, perpendicular to both layers. In the rotationally symmetric gauge, the lowest Landau level (LLL) eigenstates of an electron with the coordinate $z=x+iy$ in the plane and localized in the layer $\sigma\in\{\uparrow,\downarrow\}$ are given by

$$z^m \exp\{-|z|^2/4l_B^2\}\eta_\sigma, \quad m=0,\ldots,N_\phi-1, \tag{1}$$

where η_σ is the usual spinor wave function and the unit of length is given by the magnetic length, $l_B=\sqrt{\hbar c/eB}$. The number of flux quanta, N_ϕ, denotes the number of available states in the LLL. In the thermodynamic limit, the ratio of the number of electrons N_e and the number of flux quanta N_ϕ defines the filling factor $\nu=N_e/N_\phi$ and we focus on the particular case $\nu=2/5$.

The many-body interacting system of electrons is defined by the following Lagrangian density in the second quantized formulation:

$$\mathcal{L}=\sum_\sigma\left\{\Psi_\sigma^\dagger\partial_\tau\Psi_\sigma-\Psi_\sigma^\dagger\frac{(\partial_{\mathbf{r}}+e\mathbf{A})^2}{2m}\Psi_\sigma-\Psi_\sigma^\dagger\frac{\Delta_{SAS}}{2}\Psi_{-\sigma}\right.$$
$$+\frac{1}{2}\int d\mathbf{r}'\rho_\sigma(\mathbf{r})V_c^{\text{intra}}(\mathbf{r}-\mathbf{r}')\rho_\sigma(\mathbf{r}')$$
$$\left.+\frac{1}{2}\int d\mathbf{r}'\rho_\sigma(\mathbf{r})V_c^{\text{inter}}(\mathbf{r}-\mathbf{r}')\rho_{-\sigma}(\mathbf{r}')\right\}, \tag{2}$$

where Ψ_σ is the electron field which carries the pseudospin (layer) index and Δ_{SAS} denotes the tunneling term. The interaction is defined by

$$V_c^{\text{intra}}(r)=\frac{e^2}{\epsilon r} \tag{3}$$

and in general V_c^{inter} is different. When we model a quantum Hall bilayer,

$$V_c^{\text{inter}}(r) = \frac{e^2}{\epsilon\sqrt{r^2+d^2}}, \tag{4}$$

d has the meaning of the distance between two layers of two-dimensional gases and it is of the order of l_B. In the Lagrangian density Eq. (2) and the remainder of this paper we set $\hbar=c=l_B=1$. Significant insight into the physics described by the Lagrangian Eq. (2) can be obtained using first-quantized trial wave functions for its ground states.[13] In the remainder of this section we list several candidate wave functions that are expected to describe the ground state of Eq. (2) in different limits of Δ_{SAS} and d. Trial wave functions in the LLL are analytic in z variables and we will omit the omnipresent Gaussian factor for each electron as the one in Eq. (1).

In the small tunneling regime, the FQH system at $\nu=2/5$ is two component, described by the 332 Halperin state for two distinguishable species of electrons, $z_{i\sigma};\sigma=\uparrow,\downarrow;i=1,\ldots,N_e/2$

$$\Psi_{332} = \prod_{i<j}(z_{i\uparrow}-z_{j\uparrow})^3 \prod_{k<l}(z_{k\downarrow}-z_{l\downarrow})^3 \prod_{p,q}(z_{p\uparrow}-z_{q\downarrow})^2. \tag{5}$$

Due to the fact that the correlation exponents between electrons of the same layer are bigger than those between electrons of the opposite layers, we expect the wave function Eq. (5) to be more appropriate for non-zero d, e.g., in the range $d\sim l_B$. However, as it possesses the necessary symmetry properties,[11] it can be a candidate also for $d=0$. The properties of the wave function Eq. (5) were numerically verified in Ref. 14.

As the tunneling strength Δ_{SAS} is increased, the electrons find it energetically favorable to be in the superposition of two layers, $\uparrow+\downarrow$, and the system loses its two-component character. The effective single-component state is characterized by full polarization in the x-direction. At $\nu=2/5$ in the LLL, a compelling candidate for the polarized state is Jain's composite fermion (CF) state:[2]

$$\Psi_{\text{Jain}} = \mathcal{P}_{LLL}\left\{\prod_{i<j}(z_i-z_j)^2 \cdot \chi_2(\{z\})\right\}, \tag{6}$$

where $\mathcal{P}_{LLL}$ is a projector to the LLL and χ_2 represents the Slater determinant of two filled pseudo-Landau levels of CFs.[2] Note that a single index suffices to label the electron coordinates as the pseudospin index is implicitly assumed to be $\uparrow+\downarrow$.

Recent work[12] has introduced an alternative candidate for the polarized state at the filling factor $\nu=2/5$, the so-called Gaffnian state:

$$\Psi_{\text{Gaff}} = \mathcal{A}\left[\Psi_{332}\ \text{perm}\left(\frac{1}{z_\uparrow - z_\downarrow}\right)\right]. \tag{7}$$

In the notation of Eq. (7) one can think of the Gaffnian originating from the two-component 332 state with the additional pairing represented by the permanent, a determinant with plus signs.[15,16] The two-component state is made single-component under the action of the antisymmetrizer $\mathcal{A}$ between $\uparrow$ and $\downarrow$ electron coordinates. Gaffnian Eq. (7) has generated a surge of interest because in finite size (spherical) exact diagonalization it shows high overlaps with the Coulomb ground state, comparable to those of Jain's state, yet the topological properties of the two states are very different.[12] Moreover, the strong evidence for Gaffnian in numerical calculations is puzzling in view of the fact that it is a correlator of a nonunitary conformal field theory and, hence, not expected to describe a stable phase.[17] In the spherical geometry, Jain's state and the Gaffnian can only be distinguished by their excitation spectrum[18] or by using advanced tools such as the entanglement spectrum.[19]

Since the antisymmetrizer $\mathcal{A}$ can, to some extent, be mimicked by the tunneling term,[20] and since the Gaffnian incorporates the pairing defined by the permanent, there is an additional natural candidate which we refer to as the permanent state,

$$\Psi_{\text{perm}} = \Psi_{332}\ \text{perm}\left\{\frac{1}{z_\uparrow - z_\downarrow}\right\}. \tag{8}$$

This state distinguishes between $\uparrow$ and $\downarrow$ electrons, hence it is expected in the limit of intermediate tunneling Δ_{SAS} before a full x polarization has been achieved. Like the Gaffnian, the state Eq. (8) is related to a nonunitary conformal field theory[21] and one may expect that it plays a role of the critical state in the transition region before full x polarization.

In the following section, we study numerically the transitions between two-component and one-component states at the filling factor $\nu=2/5$ via tunneling Δ_{SAS}. We use the exact diagonalization in the spherical and torus geometries to gain complete insight into topological properties of the different competing trial states introduced here.

III. EXACT DIAGONALIZATIONS

We consider the transition from the 332 (two-component) Halperin state to the one-component state at $\nu=2/5$ via tunneling. The one-component state is identified below as Jain's (Abelian) state Eq. (6), though it is not at the same shift on the sphere as the 332 state.[10,11] The shift $\delta=N_e/\nu-N_\phi$ is a topological number[1,17] and defined through a relation between N_e and N_ϕ that is necessary for the appearance of a particular FQH state on the sphere. For example, in case of the 332 state $\delta=3$, whereas for the states in Eqs. (6)–(8) $\delta=4$. This mismatch is an unfortunate feature of the spherical geometry which prevents the direct study of the transition. However, all of the mentioned states describe the filling $\nu=2/5$ and therefore occur in the same Hilbert space under the periodic boundary conditions where the shift is trivially zero.[11,22] By that, the phases in the torus geometry do not "loose" the topological number connected with the shift on the sphere, this number that reflects the orbital spin can be characterized by the Hall viscosity of the system.[17] Thus, in the torus geometry we can study the transitions in a direct manner. As we mention below, another advantage of the torus geometry is the specific ground-state degeneracy which can be used as a fingerprint of a phase. The physical results derived from the two geometries, however, ought to agree for large enough systems. Our numerical studies are restricted to a small number of electrons because the tunneling does not conserve particle number in each layer. Since, we anticipate

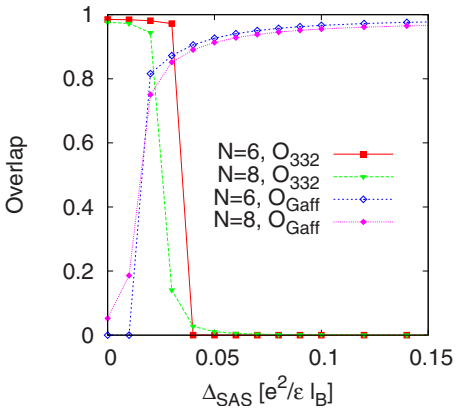

FIG. 1. (Color online) Overlaps between the exact Coulomb bilayer ground state for $d=l_B$ and the 332 (O_{332}) and Gaffnian state (O_{Gaff}), as a function of tunneling Δ_{SAS}. Data shown is for $N_e=6$ and 8 electrons. Note that O_{332} and O_{Gaff} can not be directly compared due to the difference in shift between the 332 state and the Gaffnian.

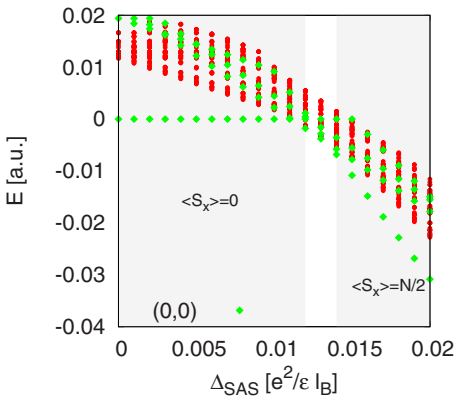

FIG. 2. (Color online) Energy spectrum of the SU(2)-symmetric 332 Hamiltonian on torus (in arbitrary units) for $N_e=8$ and aspect ratio 0.97. The $\mathbf{k}=0$ levels that cross define regions of fully polarized $\langle S_x\rangle=N/2$ and unpolarized $\langle S_x\rangle=0$ phases.

incompressible states for most of the range of Δ_{SAS}, small system sizes are nonetheless expected to be relevant as usual in the context of quantum Hall effect.[11]

A. Sphere

In the spherical geometry, Coulomb or any interaction that depends on the distance between two electrons is parameterized by a discrete series of the so-called pseudopotentials in the LLL.[11] Each pseudopotential is an eigenvalue of the interaction strength corresponding to the state of definite relative angular momentum (l) of two electrons. Therefore a series of pseudopotentials $\{V_l|l=0,1,\ldots\}$ completely specifies the interaction in the LLL. Model pseudopotentials define an interaction in the LLL for which the analytic functions of some simple fractional quantum Hall states are the densest zero energy eigenstates. This is the case for the 332 state when $V^{\mathrm{intra}}=\{0,V_1^a,0,0,\ldots\}$ and $V^{\mathrm{inter}}=\{V_0,V_1^e,0,0,\ldots\}$. There is some freedom in choosing $V_0,V_1^{a,b}$ apart from the requirement that they should all be positive and we set them to unity. Values of $V_0,V_1^{a,b}$ control the gap for the 332 state and thereby affect the critical value for the tunneling Δ_{SAS} in the following discussion, but our main conclusions remain unaffected by this choice. In the case of the Jain state we do not have a pseudopotential formulation (a useful ansatz[12] that does not lead to a unique zero-energy eigenstate is $\{0,V_1,0,0,\ldots\}$).

In Fig. 1, we present our results for the case of the bilayer Coulomb interaction on the sphere with the bilayer distance d equal to l_B. Overlaps of the exact state with the 332 state and the Gaffnian are calculated as a function of tunneling Δ_{SAS}. Separate diagonalizations have been performed because the two trial states, 332 and Gaffnian, occur in slightly different Hilbert spaces due to the mismatch in shift ($\delta=3$ and $\delta=4$, respectively). Following the rapid destruction of the 332 state with the increase of Δ_{SAS}, the overlap with the Gaffnian state rises to the high value known from earlier studies in a single-layer model.[12,19] This occurs at the point when the system is almost fully x polarized. Consequently, the overlap with the Jain state for large Δ_{SAS} is also high and virtually indistinguishable from that of the Gaffnian on the scale of this figure.

B. Torus

In the torus geometry, Figs. 2–6, trial states which describe the same filling factor $\nu=p/q$ can be directly compared because the shift is zero. What is then characteristic of the Abelian states such as the 332 and Jain's state, is that on the torus they only posses the ground state degeneracy due to the motion of the center of mass of the system, equal to q.[7] This is a trivial degeneracy and we will mode out its presence in the data. In the case of Gaffnian the degeneracy of the ground state is expected[12,23] to be doubled with respect to the trivial one, i.e., equal to $2\times 5=10$. In the literature there is no consensus that Gaffnian is a gapless state,[12,18] but if we can establish that the nature of the lowest lying states is as expected for the Gaffnian, we could nonetheless claim its presence at the transition from the 332 to the Jain's state.

In Fig. 2, we plot the low energy spectrum of the 332 short-range Hamiltonian (Sec. III A) on the torus for N

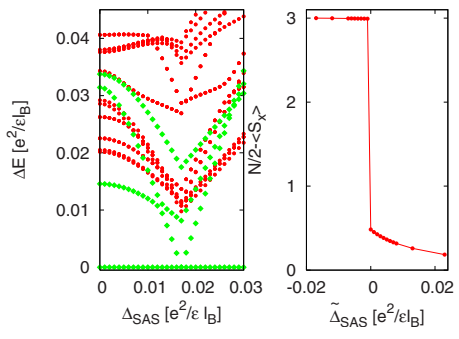

FIG. 3. (Color online) Energy spectrum relative to the ground state of the Coulomb bilayer on torus, for $N_e=6$ electrons, $d=l_B$ and aspect ratio 0.97 (left). We indicate the states characterized by $\mathbf{k}=(0,0)$ Haldane pseudomomenta. Also shown is the polarization $N/2-\langle S_x\rangle$ as a function of tunneling around the transition point $\Delta_{SAS}^C\approx 0.017e^2/\epsilon l_B$ (right).

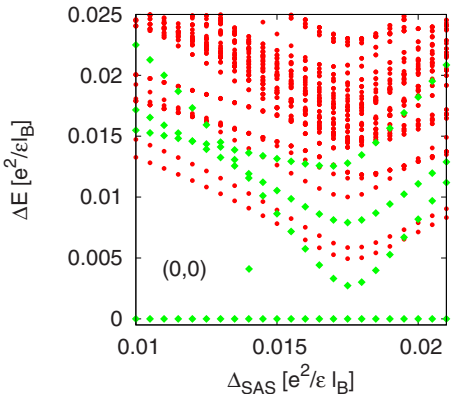

FIG. 4. (Color online) Energy spectrum relative to the ground state of the Coulomb bilayer on torus, for $N_e=8$ electrons, $d=l_B$ and aspect ratio 0.97. An approximate doublet of states with $\mathbf{k}=(0,0)$ Haldane pseudomomenta is formed around the transition point $\Delta_{SAS}\approx 0.018e^2/\epsilon l_B$.

=8 electrons and close to the square unit cell (aspect ratio $a/b=0.97$). We observe the 332 state, distinctly marked by its zero energy, which remains unaffected by Δ_{SAS} until level crossing is induced with the excited polarized state. We also calculate the mean value of the S_x projection of pseudo-spin which plays the role of the "order parameter" and has previously been used to detect the transition between quantum Hall phases.[24–26] The state characterized by $\langle S_x\rangle=N/2$ that becomes the ground state for large tunneling develops into a Jain CF state, Eq. (6). This is expected because the original system defined in terms of $V_c^{\text{intra}}(r), V_c^{\text{inter}}(r)$ (3, 4), in the limit of very large tunneling becomes an effective one-component model with the modified interaction $[V_c^{\text{intra}}(r)+V_c^{\text{inter}}(r)]/2$.[20] For the short-range 332 Hamiltonian, this is simply a V_1 pseudopotential which yields a good approximation to Jain's state.[12] Furthermore, as we vary the aspect ratio of the torus, we find the following thin torus configuration…01001…, which is that of the Jain state.[27]

Coulomb interaction shows stronger finite size effects that we exemplify with the spectra for $N=6$ (Fig. 3) and 8 electrons (Fig. 4). In these calculations, we tune the aspect ratio to the same value of $a/b=0.97$ (slightly away from unity to

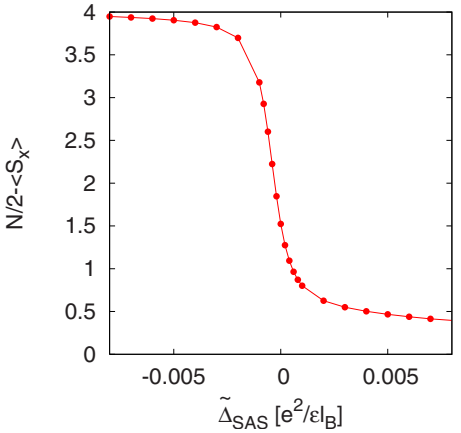

FIG. 5. (Color online) Polarization $N/2-\langle S_x\rangle$ as a function of tunneling around the transition point for the system of Fig. 4.

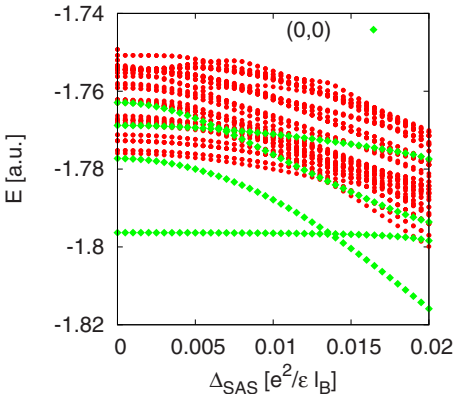

FIG. 6. (Color online) Energy spectrum of the Coulomb bilayer on torus (in arbitrary units) for $N_e=8$ and aspect ratio 0.5.

avoid accidental geometric degeneracy) and distance between layers is set to $d=l_B$. The incompressible states for small and large tunneling in Fig. 3 can be identified as the 332 and the Jain state, with the transition between them occurring for $\Delta_{SAS}^C\approx 0.017e^2/\epsilon l_B$ when the levels cross (Fig. 3, right), suggestive of the first order transition. As a consequence, the polarization ("order parameter") $N/2-\langle S_x\rangle$ experiences a sharp discontinuity at the point of transition (Fig. 3, left). We stress that this level crossing occurs for a wide range of aspect ratios of the torus and not only in the vicinity of the square unit cell.

On the other hand, for the larger system of $N_e=8$ electrons interacting with Coulomb interaction, we obtain the transition that proceeds via level repulsion instead of level crossing, Fig. 4. We again identify incompressible states for small and large tunneling as the 332 and the Jain state, with the transition between them occurring for $\Delta_{SAS}^C\approx 0.018e^2/\epsilon l_B$. The states can be identified, e.g., with respect to the Fig. 2 by calculating overlaps. If we denote the ground state of the short-range and Coulomb Hamiltonian for a given tunneling Δ_{SAS} as $\Psi_{\text{short}}(\Delta_{SAS})$ and $\Psi_C(\Delta_{SAS})$, respectively, we obtain the following overlap $\langle\Psi_{332}|\Psi_C(\Delta_{SAS})\rangle \equiv\langle\Psi_{\text{short}}(\Delta_{SAS}=0)|\Psi_C(\Delta_{SAS}=0)\rangle\approx 0.95$. This means that the Coulomb bilayer ground state is nearly the same as the 332 state, assuming zero tunneling. Also, in the large tunneling limit, we obtain e.g. $\langle\Psi_{\text{Jain}}|\Psi_C(\Delta_{SAS}\to\infty)\rangle\approx\langle\Psi_{\text{short}}(\Delta_{SAS}\to\infty)|\Psi_C(\Delta_{SAS}\to\infty)\rangle\approx 0.948$, i.e. Jain's state. The quantity which describes the density of the odd channel, $N/2-\langle S_x\rangle$, characterizes the transition by an approximately linear or even steplike discontinuity as a function of $\tilde{\Delta}_{SAS}=\Delta_{SAS}-\Delta_{SAS}^C$, Fig. 5. In the transition region, an approximate doublet of states with $\mathbf{k}=0$ Haldane pseudomomenta is formed (Fig. 4). Although the doublet has the expected quantum numbers of the Gaffnian,[23] the specific root configurations in the thin torus limit[27] cannot be unambiguously identified as those of the Gaffnian. Both of the members of the doublet share the following thin torus configuration…01001…, among other spurious patterns, which is that of the Jain state. Moreover, the member of the doublet higher in energy has a lower polarization $\langle S_x\rangle$ than the ground state. These facts suggest that the excited $\mathbf{k}=0$ state in the transition region is a spinful CF state rather than the (polarized) Gaffnian.

For the long-range $N=8$ Coulomb system on the torus and the aspect ratio close to 1, the transition between the Jain and

332 state proceeds as an avoided level crossing or a smooth crossover without an obvious closing of the gap. The gap is expected to close in the thermodynamic limit between the two distinct topological phases, although we are unable to perform a proper finite-size scaling of the gap due to the inaccessibility of the $N=10$ electron system. However, for the short-range interaction that defines the 332 state as the zero-energy ground state and for the identical geometry of the torus ($a/b=0.97$), it appears that the gap indeed closes, Fig. 2. This difference between Figs. 2 and 4 can be attributed to the symmetry of the interaction. For the short range interaction used in Fig. 2, $V_1^{\text{inter}}=V_1^{\text{intra}}$, hence it does not break the SU(2) symmetry. In this case, the tunneling part of the total Hamiltonian, being proportional to S_x component, commutes with the interaction part and we expect level crossing which we indeed observe in Fig. 2. The interaction in the bilayer with $d=l_B$, on the other hand, breaks SU(2) invariance (Fig. 4), but we can nevertheless show that the level crossing persists and can be induced by changing the aspect ratio of the torus away from unity. In Fig. 6, we show one such energy spectrum (without the ground state energy subtraction) when the aspect ratio is equal to 0.5. The level crossing is induced by deforming the system towards the crystalline limit, when the Coulomb interaction is increasingly of short range. Note, however, that the states at $\Delta_{SAS}=0$ and Δ_{SAS}-large are still 332 and Jain's, respectively (verified by the overlaps with the ground state of the short-range interaction and by their thin torus limit).

IV. EFFECTIVE BOSONIC MODEL

A. Introduction

High overlaps with the Gaffnian on the sphere around and after the transition are a motivation for considering the system of Chern-Simons (CS) transformed composite bosons[28,29] ($\uparrow$ and $\downarrow$) that pair in the way of p-wave in a picture of the underlying neutral sector physics. This bosonic system is, by its very nature, unstable towards the ordinary Bose condensation, as shown for the first time in Ref. 30, and, as we will elaborate more, the pairing may be realized only in its excited states or at a transition point. As we will discuss in this Section, a simple underlying CS bosonic picture of the 332 and Jain's state will be enlarged by p-wave fluctuations. The fluctuations are expected to play a role near the transition and in the description of the critical and excited states, but not in the well-developed phases-the ground states away from the transition. As we already pointed out in the preceding section, the high Gaffnian overlaps are not to be taken as a proof that we have the Gaffnian phase after the transition, in the thermodynamic limit, but may serve as a motivation for discussing the role for the Gaffnian as a critical state. More generally, as the system is closer to the one-component limit, the theory may inherit the pairing structure built in the Gaffnian state and this is captured in the permanent state, Eq. (8). As we mentioned in Sec. II, the connection between the Gaffnian Eq. (7) and the permanent state (a p-wave state of bosons) Eq. (8) is the antisymmetrization. We assume that the operation of antisymmetrization corresponds, in the language of effective theory, to a tunneling term.[20]

B. Bosonic model

To begin with, one may perform CS transformations in the field-theoretical description of the system Eq. (2) that would leave, in the mean field, $\uparrow$ and $\downarrow$ bosons that pair in the way of a p-wave. At $\nu=2/5$, for no tunneling, in the presence of Coulomb or suitable short range interaction, we expect that the bilayer (two-component) system is described by 332 state. We know very well how to define the CS transformation to bosons in these circumstances, for the first time it was given in Ref. 29. It entails a transformation from electronic Ψ_σ fields [in Eq. (2) with $\Delta_{SAS}=0$] to bosonic Φ_σ fields:

$$\Psi_\sigma(\mathbf{r}) = U_\sigma(\mathbf{r})\Phi_\sigma(\mathbf{r}), \tag{9}$$

where

$$U_\sigma(\mathbf{r}) = \exp\left\{-i\int d\mathbf{r}'\ \arg(\mathbf{r}-\mathbf{r}')[3\rho_\sigma(\mathbf{r}')+2\rho_{-\sigma}(\mathbf{r}')]\right\}, \tag{10}$$

where $\arg(\mathbf{r}-\mathbf{r}')$ is the angle the vector $\mathbf{r}-\mathbf{r}'$ forms with the x axis. In the mean field (when the fluctuations of gauge fields are neglected) we, in fact, describe a system of $\uparrow$ and $\downarrow$ bosons that interact. Therefore, we have in the first approximation two ordinary Bose condensates. By the virtue of the Anderson-Higgs mechanism i.e. gauge fluctuations, the two Goldstone modes become gapped and the two gapped bosonic systems describe the two-component 332 system.

The complication comes when we consider the tunneling term as an extra perturbation and an extra term in our starting Hamiltonian for the electrons. The tunneling term is

$$H_T = -\lambda[\Psi_\uparrow^\dagger(\mathbf{r})\Psi_\downarrow(\mathbf{r}) + \Psi_\downarrow^\dagger(\mathbf{r})\Psi_\uparrow(\mathbf{r})], \tag{11}$$

where λ denotes the tunneling amplitude in this section. Due to the CS transformation Eq. (9) this can not be translated simply into the hopping of bosons because:

$$\Psi_\sigma^\dagger\Psi_{-\sigma} = \Phi_\sigma^\dagger U_\sigma^\dagger U_{-\sigma}\Phi_{-\sigma} \tag{12}$$

and only in the mean-field approximation for which

$$U_\sigma^\dagger U_{-\sigma} \approx I \tag{13}$$

(where I is the identity) we have a simple tunneling of bosons i.e.

$$H_T \approx -\lambda(\Phi_\uparrow^\dagger(\mathbf{r})\Phi_\downarrow(\mathbf{r}) + \text{H.c.}). \tag{14}$$

The necessary assumption in Eq. (13) is $\rho_s=\rho_\uparrow(\mathbf{r})-\rho_\downarrow(\mathbf{r})\approx 0$ i.e. that the fluctuations in density in $\uparrow$ pseudospin parallel the ones in $\downarrow$ or the fluctuations in the pseudospin density are negligible.

Treating the residual interaction in a mean field manner, i.e., taking the Hartree-Fock and BCS decomposition, we come to the following form of the Hamiltonian for the effective description of $\uparrow$ and $\downarrow$ bosons around the $\mathbf{k}=0$ point in the momentum space:

$$H = \sum_{\mathbf{k}} \left(\sum_{\sigma} e\hat{b}^{\dagger}_{\mathbf{k}\sigma}\hat{b}_{\mathbf{k}\sigma} - \lambda(\hat{b}^{\dagger}_{\mathbf{k}\uparrow}\hat{b}_{\mathbf{k}\downarrow} + \hat{b}^{\dagger}_{\mathbf{k}\downarrow}\hat{b}_{\mathbf{k}\uparrow}) + d\hat{b}^{\dagger}_{\mathbf{k}\uparrow}\hat{b}^{\dagger}_{-\mathbf{k}\downarrow} + c\hat{b}_{-\mathbf{k}\downarrow}\hat{b}_{\mathbf{k}\uparrow} \right), \tag{15}$$

where $e=\epsilon_{\mathbf{k}}-\mu$ and $d=c^*$ is the p-wave order parameter function $d\sim k_x-ik_y$. The question of mutual statistics (between $\uparrow$ and $\downarrow$ electrons and the ensuing composite bosons) may be raised but we will assume that it is bosonic.

The Bogoliubov equations, $[\alpha_{\mathbf{k}},H]=E\alpha_{\mathbf{k}}$, where

$$\alpha_{\mathbf{k}} = u_{\uparrow}\hat{b}_{\mathbf{k}\uparrow} + u_{\downarrow}\hat{b}_{\mathbf{k}\downarrow} + v_{\uparrow}\hat{b}^{\dagger}_{-\mathbf{k}\uparrow} + v_{\downarrow}\hat{b}^{\dagger}_{-\mathbf{k}\downarrow} \tag{16}$$

define the following matrix

$$\begin{bmatrix} e & -\lambda & 0 & -c \\ -\lambda & e & c & 0 \\ 0 & -d & -e & \lambda \\ d & 0 & \lambda & -e \end{bmatrix}$$

for the eigenvalue problem. There are two pairs of eigenvalues:

$$E+\lambda, E-\lambda, \quad \text{and} -E+\lambda, -E-\lambda, \tag{17}$$

where $E=\sqrt{e^2-\Delta^2}, \Delta^2=dc$, with the corresponding unnormalized eigenvectors:

$$\alpha^{+}_{\mathbf{k}} = [a,-a,1,1], \quad \beta^{+}_{\mathbf{k}} = [a,a,-1,1], \text{and} \tag{18}$$

$$\alpha^{-}_{\mathbf{k}} = [b,-b,1,1], \quad \beta^{-}_{\mathbf{k}} = [b,b,-1,1], \tag{19}$$

where $a=\frac{e+E}{d}$ and $b=\frac{e-E}{d}$.

C. Bose condensate solution

The last two eigenvalues $-E\pm\lambda$ and vectors $\alpha^{-}_{\mathbf{k}},\beta^{-}_{\mathbf{k}}$ Eq. (19) define a pair of solutions in the form of Eq. (16) and represent well-defined excitations of the system. The ground state can be expressed as

$$\exp\left\{ \sum \frac{1}{b}\hat{b}^{\dagger}_{\mathbf{k}e}\hat{b}^{\dagger}_{-\mathbf{k}o} \right\}|0\rangle, \tag{20}$$

where $\hat{b}_{\mathbf{k}e}=\hat{b}_{\mathbf{k}\uparrow}+\hat{b}_{\mathbf{k}\downarrow}$ and $\hat{b}_{\mathbf{k}o}=\hat{b}_{\mathbf{k}\uparrow}-\hat{b}_{\mathbf{k}\downarrow}$. We have for $\mu>0$:

$$-E\pm\lambda \approx -\mu + \epsilon_{\mathbf{k}} + \frac{\Delta^2}{2\mu} \pm \lambda, \tag{21}$$

i.e., ordinary noninteracting boson description where the tunneling λ defines the transition at $\lambda=\mu$ from the two Bose condensates to a one Bose condensate (one disappears because $\mu^{\text{eff}}=\mu-\lambda<0$ i.e. we have vacuum for these particles). The mean field ground state in the $\mathbf{k}\to 0$ limit is approximately constant $(1/b\sim d\to 0)$ as it should be for the effective description of the system with (two-becoming one) Bose condensates.

This simple system in the presence of a short-ranged interaction, in the channel that changes the sign of the effective chemical potential, is described in Chapter 11.3 of Ref. 31. There $d_{\text{spatial}}=2$ was identified as an upper critical dimension. Therefore we might expect in that case, with an interaction, that the density of bosons in this channel for $\lambda<\lambda_c=\mu$ vanishes linearly with $\mu-\lambda=\mu^{\text{eff}}$ as $\lambda\to\lambda_c$.

The quantum Hall system as a whole, together with the CS fluctuations, may experience a transition with the Bose condensates becoming gapped via Anderson-Higgs mechanism(s) away from the transition.

This analogy also motivates to consider that a viable composite boson effective description of the $\nu=2/5$ Jain's state is with only one composite boson condensate and a Bose vacuum. This comes as a natural consequence from our analysis and the multicomponent approach to Jain's states.[16] From the results of the experiments on the edge of FQH states,[32] it is justified to assume an existence of a single charge mode that stems from a single Bose condensate in an effective description.

D. Role of the permanent

The transition may be discussed considering also the other pair of eigenvalues from the eigenvalue problem:

$$E\pm\lambda \tag{22}$$

and the excitations that they define Eq. (18). It is obvious from Eq. (21) that they are unstable and may describe excited states. As particular solutions of the Bogoliubov equations for the Hamiltonian defined in Eq. (15), the solutions described by $E\pm\lambda$ and their corresponding eigenvectors Eq. (18) are nonunitary and nonphysical because they are related to the physical solutions Eq. (19) by the following nonunitary relationships: $(\alpha^{-}_{-\mathbf{k}})^{\dagger}=i\beta^{+}_{\mathbf{k}}$ and $(\beta^{-}_{-\mathbf{k}})^{\dagger}=-i\alpha^{+}_{\mathbf{k}}$ which imply: $[(\beta^{+}_{\mathbf{k}})^{\dagger},\beta^{+}_{\mathbf{k}}]=[(\alpha^{+}_{\mathbf{k}})^{\dagger},\alpha^{+}_{\mathbf{k}}]=1$. We assume a possibility that the description of the system is given by H and an additional term[20] λN, i.e., $H+\lambda N$, where N is the number of particles. This term is of a purely phenomenological origin; it is designed to regularize the behavior at large λ [compare with Eq. (24)]. It can be incorporated in the previous description by a simple redefinition of $e=\epsilon_{\mathbf{k}}-\mu$ into $e=\epsilon_{\mathbf{k}}-\mu+\lambda$. This yields

$$E = \sqrt{(\epsilon_{\mathbf{k}}-\mu+\lambda)^2-\Delta^2} = |\mu-\lambda|\sqrt{1-\frac{(\mu-\lambda)}{(\mu-\lambda)^2}2\epsilon_{\mathbf{k}}-\frac{\Delta^2}{(\mu-\lambda)^2}}, \tag{23}$$

and for large tunneling $\lambda>\mu$ we have

$$E \approx \lambda-\mu+\epsilon_{\mathbf{k}}-\frac{1}{2}\frac{\Delta^2}{(\lambda-\mu)}. \tag{24}$$

The excitations Eq. (22) become $E-\lambda\approx\epsilon_{\mathbf{k}}-\mu$ and $E+\lambda=\epsilon_{\mathbf{k}}+2\lambda-\mu$, and we obtain Bose condensation in one channel and Bose vacuum in the other, as in the case of Sec. IV C. Here, for $\lambda<\mu$ we allow the possibility that by this formalism we can describe an excited state of the system which is given by the Bogoliubov expression:

$$\exp\left\{ \sum \frac{1}{a}\hat{b}^{\dagger}_{\mathbf{k}e}\hat{b}^{\dagger}_{-\mathbf{k}o} \right\}|0\rangle \tag{25}$$

and because $\frac{1}{a}=\frac{e-E}{c}$, and $\Sigma\frac{1}{a}\hat{b}^{\dagger}_{e\mathbf{k}}\hat{b}^{\dagger}_{o-\mathbf{k}}\sim\Sigma\frac{1}{a}\hat{b}^{\dagger}_{\uparrow\mathbf{k}}\hat{b}^{\dagger}_{\downarrow-\mathbf{k}}$ we have a p-wave paired permanent state in the long distance limit.

This must be an excited state because the excitations Eq. (22) are unstable for $\lambda < \mu$

$$E \approx \mu - \lambda - \epsilon_k - \frac{1}{2}\frac{\Delta^2}{(\mu - \lambda)}. \tag{26}$$

At the transition $\lambda = \mu$ we have

$$E \approx \pm i\Delta_0 |k|, \tag{27}$$

where Δ_0 is defined by $\Delta^2 = \Delta_0^2 k^2$. This defines a nonunitary system with complex values for the excitations $E \pm \lambda$. If we neglect the presence of λ for a moment, we can describe this system by a $2+1$ dimensional theory for bosons β and γ with the following Hamiltonian:

$$H = \gamma \partial_x \beta. \tag{28}$$

We quantize the system in the following manner:

$$\gamma = \sum (\exp\{-ikx\} b_k + \exp\{ikx\} a_k^\dagger)$$

$$\beta = \sum (-\exp\{ikx\} b_k^\dagger + \exp\{-ikx\} a_k) \tag{29}$$

and this reproduces the spectrum we have for $\lambda = 0$. This system is closely related to the $\beta - \gamma$ ghost system in $1+1$ dimension or the CFT connected with permanent state[21] and more generally Gaffnian[12] in its two-component formulation Eq. (7). The complete spectrum is reproduced by $H = \gamma \partial_x \beta + \lambda(\gamma\beta)$.

Therefore before reaching the strong tunneling limit and the incompressible FQH state connected with the single BCS condensate in this description at $\nu = 2/5$ (Jain's state), we may find a state at the transition that evolves from an excited state. The excited state above the 332 ground state Eq. (25) is described in the long-distance limit by a permanent times the Abelian 332 factor, Eq. (8). We note that the permanent state carries the maximum pseudospin $[S = N_e/2, \mathbf{S}^2 = S(S+1)]$, because only such states (with also $S_z = 0$) can be antisymmetrized completely in the coordinate space and make a polarized electronic wave function just as in the case of the permanent state and the ensuing Gaffnian wave function. The 332 state, on the other hand, cannot be antisymmetrized,[16] because it is a spin-singlet ($S = 0$). Depending on the ground state evolution, the polarization of the system ($\langle S_x \rangle$) may either experience a jump across the transition or the ground state may evolve smoothly into a $(\uparrow + \downarrow)$ polarized state. The state at the transition in the second case might be Gaffnian—its description in the BCS formulation is that of a state which evolves from the permanent under the effect of tunneling which may mimic the antisymmetrization as in Eq. (7). But our analysis above (Eq. (25) with the redefined e and Eq. (27)) shows that the system at the transition is still unpolarized and cannot describe the Gaffnian.

E. Discussion

According to our numerical results in Figs. 3–5 in the presence of Coulomb interaction a possible scenario is the scenario described in the Sec. IV C with effectively one of the two Bose condensates disappearing with the increase of tunneling. If we include interactions in the simple bosonic model they can smooth the transition (compare with results in Figs. 4 and 5). In Fig. 5, we see the linear dependence of the number of odd channel electrons on the tunneling strength near the transition. In the $\mathbf{k} \to 0$ limit the density of the odd channel is equal to the density of the vanishing Bose condensate. Therefore this linear dependence may stem from the critical behavior of dilute bosons as described in Ref. 31. $d_{\text{spatial}} = 2$ is the upper critical dimension in this case and we may expect a logarithmic correction to the linear behavior as demonstrated in Ref. 33, in the case of a short range interaction among bosons. In our case Coulomb interaction may be driving the fixed point for the short range interactions into a mean field one with linear behavior. We calculated the density-density correlator in the transition region, but definite conclusion about the power of the decay of the correlations could not be drawn because of the finite size effects. A lower bound for the exponent that governs the decay with the distance is equal to 2, as expected in the mean field.

Thus the bosonic model with interactions may lead to a second-order transition with gradually disappearing bosons. In a more elaborate description one may hope that Gaffnian will appear as a polarized critical state before the polarized Jain state. But if the state at the transition is partially polarized, as we find in exact diagonalizations and effective bosonic model (without repulsive interactions), even in the Coulomb case we may expect a first-order transition or a smooth crossover without Gaffnian.

In the following, we discuss implications of our analysis for the effective bosonic description of the Jain's $\nu = 2/5$ state. If, due to tunneling, one Bose condensate indeed vanishes, the effective description would then comprise only one Bose condensate and a Bose vacuum. On the other hand any effective description of quantum Hall states must encompass the edge physics as the low-energy physics of these states happens on the edge. In the effective description based on the usual picture with composite bosons[1] of the $\nu = 2/5$ fractional quantum Hall edge, both charge and neutral edge modes of two condensates propagate in the same direction as relativistic particles and the discrepancy with respect to experiments,[32] which detect only one (charge) mode, has to be resolved.[34] In the effective description based on composite fermions,[35] at $\nu = 2/5$ edge only the charge mode is propagating, in agreement with the experiment, but the reason why the neutral mode does not propagate is not obvious. Here, we suggest an effective picture of the neutral i.e. multicomponent degrees of freedom of Jain's state at $\nu = 2/5$ via a Bose vacuum. An edge excitation of the system that involves also these, multicomponent, degrees of freedom, is accompanied by a bosonic excitation of a vacuum that propagates, not relativistically, but according to Schrödinger equation,[31] which in an effective description for certain probes can be neglected with respect to the charge wave propagation along the edge.

V. CONCLUSIONS

We studied, by numerical and analytical means, the transition from the two-component to a one-component quantum

Hall state induced by tunneling at the filling factor $\nu=2/5$. The transition is studied in the presence of the Coulomb interactions appropriate for a quantum Hall bilayer and a model short-ranged interaction appropriate for the 332 Halperin's state. In exact diagonalizations of small systems two possibilities for the transition are found: (a) avoided level crossing, and (b) level crossing, i.e., first-order transition in the case of the Coulomb interaction and short range interaction, respectively.

With respect to the appearance of the Gaffnian state in the transition region between 332 and Jain state, we can conclude that in finite systems this is only possible for the interaction that breaks SU(2) invariance, like the Coulomb bilayer interaction. It is an unlikely possibility, however, even for non-SU(2) invariant interaction, because of the difficulty in establishing the thin torus limit for the approximate $\mathbf{k}=0$ doublet found for the torus with aspect ratio close to unity-(Fig. 4). In other words, on the thin torus we observe only a "half"[12] of the Gaffnian physics that corresponds to Jain's state. So long as the interaction is nearly SU(2) invariant, the transition occurs via level crossing (Figs. 2 and 6) and it is a first-order transition between the unpolarized and polarized Abelian states.

Also, to probe the question of p-wave pairing and related Gaffnian correlations at the transition we introduced an effective bosonic model. We find that the transition in the presence of the Coulomb interaction may be viewed as a transition from two Bose condensates to a Bose condensate and a Bose vacuum. The outcome, with the Bose vacuum, can serve as an effective description of the Jain state. In the simple bosonic picture we find that the state at the transition does not correspond to the polarized Gaffnian state, in accordance with the exact diagonalizations.

ACKNOWLEDGMENTS

We thank Mark Goerbig and Nicolas Regnault for discussions. This work was supported by the Serbian Ministry of Science under Grant No. 141035.

[1] X.-G. Wen, Int. J. Mod. Phys. B **6**, 1711 (1992).
[2] J. Jain, *Composite Fermions* (Cambridge University Press, Cambridge, England, 2007).
[3] B. I. Halperin, Helv. Phys. Acta **56**, 75 (1983).
[4] X.-G. Wen, Phys. Rev. Lett. **84**, 3950 (2000).
[5] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
[6] I. A. McDonald and F. D. M. Haldane, Phys. Rev. B **53**, 15845 (1996).
[7] F. D. M. Haldane, Phys. Rev. Lett. **55**, 2095 (1985).
[8] X.-G. Wen and A. Zee, Phys. Rev. B **46**, 2290 (1992).
[9] X.-G. Wen, Phys. Rev. B **40**, 7387 (1989).
[10] X. G. Wen and A. Zee, Phys. Rev. Lett. **69**, 953 (1992).
[11] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983); See also *The Quantum Hall Effect*, 2nd ed., edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1990).
[12] Steven H. Simon, E. H. Rezayi, N. R. Cooper, and I. Berdnikov, Phys. Rev. B **75**, 075317 (2007).
[13] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[14] D. Yoshioka, A. H. MacDonald, and S. M. Girvin, Phys. Rev. B **39**, 1932 (1989).
[15] D. Yoshioka, A. H. MacDonald, and S. M. Girvin, Phys. Rev. B **38**, 3636 (1988).
[16] N. Regnault, M. O. Goerbig, and Th. Jolicoeur, Phys. Rev. Lett. **101**, 066803 (2008).
[17] N. Read, Phys. Rev. B **79**, 045308 (2009).
[18] C. Tőke and J. K. Jain, Phys. Rev. B **80**, 205301 (2009).
[19] N. Regnault, B. A. Bernevig, and F. D. M. Haldane, Phys. Rev. Lett. **103**, 016801 (2009).
[20] Z. Papić, M. O. Goerbig, N. Regnault, and M. V. Milovanović, arXiv:0912.3103 (unpublished).
[21] N. Read and E. Rezayi, Phys. Rev. B **54**, 16864 (1996).
[22] F. D. M. Haldane and E. H. Rezayi, Phys. Rev. Lett. **54**, 237 (1985).
[23] E. Ardonne, Phys. Rev. Lett. **102**, 180401 (2009).
[24] K. Nomura and D. Yoshioka, Phys. Rev. B **66**, 153310 (2002).
[25] John Schliemann, S. M. Girvin, and A. H. MacDonald, Phys. Rev. Lett. **86**, 1849 (2001).
[26] G. Möller, S. H. Simon, and E. H. Rezayi, Phys. Rev. Lett. **101**, 176803 (2008); Phys. Rev. B **79**, 125106 (2009).
[27] E. J. Bergholtz, T. H. Hansson, M. Hermanns, A. Karlhede, and S. Viefers, Phys. Rev. B **77**, 165325 (2008).
[28] S.-C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989); S. C. Zhang, Int. J. Mod. Phys. B **6**, 25 (1992).
[29] Z. F. Ezawa and A. Iwazaki, Phys. Rev. B **48**, 15189 (1993).
[30] D. Green, arXiv:cond-mat/0202455.
[31] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).
[32] M. Grayson, D. C. Tsui, L. N. Pfeiffer, K. W. West, and A. M. Chang, Phys. Rev. Lett. **80**, 1062 (1998).
[33] Subir Sachdev, T. Senthil, and R. Shankar, Phys. Rev. B **50**, 258 (1994).
[34] D.-H. Lee and X.-G. Wen, arXiv:cond-mat/9809160 (unpublished).
[35] A. Lopez and E. Fradkin, Phys. Rev. B **59**, 15323 (1999).